كلية الهندســـة بشبرا
FACULTY OF ENGINEERING- SHOUBRA

Electromagnetic Fields

Part 2 - Lecture (4)

Magnetic Fields Part II

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$\vec{B}$

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$\vec{B}$

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Introduction

- So far the discussions are restricted to static or time-invariant electromagnetic fields. In these fields, electric and magnetic fields are independent of each other.
- Now the discussions is extended to dynamic or time-varying electric and magnetic fields. In these fields, the two fields are interdependent. In other words, a time-varying electric field necessarily involves a corresponding time-varying magnetic field.
- It has been already mentioned that electrostatic fields are due to static electric charges whereas magnetostatic fields are produced due to motion of electric charges with uniform velocity (direct current) or static magnetic charges (magnetic poles).
- The time varying fields or waves are usually due to accelerated charges or time-varying currents.
- In summary, stationary charges (q) $\rightarrow$ electrostatic fields.
- Steady currents (or) motion of charges $(dq/dt = I)$ $\rightarrow$ Magnetostatic fields.
- Time-varying current or accelerated charges $(dI/dt = d^2q/dt^2)$ $\rightarrow$ Electromagnetic fields (or) waves.

Study "2" theories For Maxwell' equation But in Magnetic / Electric Time varying.

1st equation • iF the Magnetic Flux lines which induces a wire loop is varying with time, ∴ the Induced Current will Results & due to this current:-
a potential (voltage) is Induced ∴

$$\boxed{V = -\frac{\partial \Phi}{\partial t}}$$

Induced Voltage ← V ; Φ → Magnetic flux

• EMF about closed Path is equal to Line Integral of Electric field about the path

$$\boxed{Emf = \oint_c \bar{E} \cdot d\bar{l}}$$ → electromotive force

القوة الدافعة الكهربية تعتمد
تتركز في E field
ق.د.ك $= \oint_c E \cdot dl$

- Faraday's says that $\oint \bar{E} \cdot d\bar{l}$ = Negative Rate of change of Magnetic flux density on surface

لو عندي مسار مغلق ∴ مجموع E على طول line = مجموع معدل تغير B بالنسبة للزمن على السطح

$$V_{emf} = \oint \bar{E} \cdot d\bar{l} = \iint_s -\frac{\partial B}{\partial t} \cdot d\bar{s}$$

- Now From Stock theorem

نحول التكامل الـ line لتكامل سطحي

$$\therefore \oint \bar{E} \cdot d\bar{l} = \iint_s (\nabla \times \bar{E}) \cdot ds = \iint_s -\frac{\partial \bar{B}}{\partial t} \cdot d\bar{s}$$

$$\therefore \nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

معادلة ماكسويل الأولى في Time varying

Summary Faraday's Law for Time varying Fields

- Eqn.(2.32) is called Faraday's law for Electrostatic which is rewritten as:

$$\oint_L \vec{E}.d\vec{l} = \int_S (\nabla X\vec{E}).d\vec{S} = 0$$

Integral Form

$$\nabla X\vec{E} = 0 \qquad (2.32)$$

Differential Form

This is known as Faraday's Law for Electrostatic Field

- Eqns. (4.9a) and (4.9b) are called Faraday's law for Time varying magnetic field which are rewritten as:

$$V_{emf} = \oint_L \vec{E}.d\vec{l} = -\int_S \frac{\partial\vec{B}}{\partial t}.d\vec{S} + \oint_L (\vec{u}x\vec{B}).d\vec{l} \qquad (4.8)$$

Integral Form

$$\nabla x\vec{E} = -\frac{\partial\vec{B}}{\partial t} + \nabla x(\vec{u}x\vec{B}) \qquad (4.9)$$

Differential Form

- Combine the above equations, and for nonmoving circuit, we have:

$$V_{emf} = \oint_L \vec{E}.d\vec{l} = -\int_S \frac{\partial\vec{B}}{\partial t}.d\vec{S} \qquad (4.10a)$$

Integral Form

$$\nabla x\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad (4.10b)$$

Differential Form

- In Conclusion: Eqn. (4.10) represents Faraday's Law also is one of the four Maxwell's equations in both integral and differential form.

4.3 Ampere's Law for Time varying Fields

4.3.1 Types of Current Densities

Types of Current Densities

Conduction Current Density J_{cond} (in Conductors)	Convection Current Density J_{conv} (moving charge with velocity v)	Displacement Current Density J_d (in Capacitors)

1. Conduction Current

- When an electric field is applied to a conductor, conduction current occurs due to the drift motion of electrons. As the electrons move, they encounter some damping forces called resistance. The average drift velocity of the elections is directly proportional to the applied field. Thus, conduction current density is:

$$\vec{J}_{cond} = \sigma \vec{E} \qquad (4.11)$$

where σ is the conductivity of the materials in Siemens per meter and J is known as conduction current density (in A/m^2). Eqn. (4.11) is referred to as Ohm's law in point form. The resistance of conductor of uniform cross-section, A and of length' l' is:

$$R = \frac{V}{I} = \frac{\int \vec{E}.d\vec{l}}{\int \sigma \vec{E}.d\vec{S}} = \frac{\rho_c l}{A} \quad [\Omega] \qquad (4.12)$$

where V is the potential difference in volts [V], I is the current in [A], $\rho_c = 1/\sigma$ is the resistivity of the material and R is the resistance in [Ω].

4.3 Ampere's Law for Time varying Fields (Continued)

4.3.1 Types of Current Densities

2. Convection Current

- Convection current, as distinct from conduction current, does not involve conductors and consequently does not satisfy Ohm's law.
- It occurs when current flows through an insulating medium such as liquid, vacuum etc. For example, a beam of electrons in a vacuum tube is a convection current. The current through a given area is defined as the electric charge passing through the area per unit time. That is:

$$I = \frac{dQ}{dt} \qquad (4.13)$$

- If there is a flow of charge of density p_v, in a filament at a velocity v, then current through the filament is given by eqn. (4.13) as:

$$\Delta I = \frac{\Delta Q}{\Delta t} = \frac{\rho_v \Delta v}{\Delta t} = \frac{\rho_v (\Delta S.\Delta l)}{\Delta t} = \rho_v \Delta S \frac{\Delta l}{\Delta t} = \rho_v \Delta S u \qquad (4.14)$$

where Δv is a differential volume of filament in m^3, ΔS is cross-section area and u is the beam velocity. As current density at a point is the current through the area at that point, the eqn. (4.14) can be modified to give the convection current density, J_{conv} (in A/m^2).as:

$$\vec{J}_{conv} = \frac{\Delta I}{\Delta S} = \rho_v \vec{u} \qquad (4.15)$$

4.3 Ampere's Law for Time varying Fields (Continued)

4.3.1 Types of Current Densities

3. Displacement Current

- The law of conservation of charge states that current due to the flow of charge out of a closed surface, 'S' bounding a volume 'v' must be equal to the time rate of decrease of charge enclosed by the surface.
- If the current flowing out of the surface is $[I_c]_S$ and the charge enclosed by the surface is Q coulomb, then:

$$[I_C]_S = -\frac{dQ}{dt} \qquad (4.16)$$

- In terms of current density J, charge density p_v and Q (2.16) can be written as:

$$\oint_S \vec{J}.d\vec{S} = -\frac{d}{dt}\int_v \rho_v dv \qquad (4.17a)$$

- Applying divergence theorem to the left-hand side and interchanging the differentiation and integrating on the right side, eqn. (2.17) becomes:

$$\int_v (\nabla.\vec{J}).dv = -\frac{\partial}{\partial t}\int_v \rho_v dv \quad \Rightarrow \quad \nabla.\vec{J} = -\frac{\partial}{\partial t}\rho_v \qquad (4.17b)$$

- Eqn. (2.17a) is called continuity equation (or) the law of conservation of charge in differential form. Eqn. (2.17b) is called law of conservation charge in integral form.

4.3 Ampere's Law for Time varying Fields (Continued)

4.3.1 Types of Current Densities

3. Displacement Current

- In the previous section, the equations for electrostatic fields are modified for time-varying situations and also to satisfy Faraday's law, Again, the equation for magnetostatic fields is to be modified for time-varying condition.
- For electrostatic magnetic fields, Ampere circuital law states that:
- When the magnetic field is time varying it generates a time varying electric field that generates a displacement current as shown in Fig. 4.3. Ampere's Circuital Law in point form therefore becomes:

$$\nabla x\vec{H} = \vec{J} + \vec{J}_d = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad (4.18a)$$

Where

$$\vec{J}_d = \frac{\partial \vec{D}}{\partial t} \qquad (4.18b)$$

- The term $\vec{J}_d$ is known as displacement current density and $\vec{J}$ is the conduction current density.

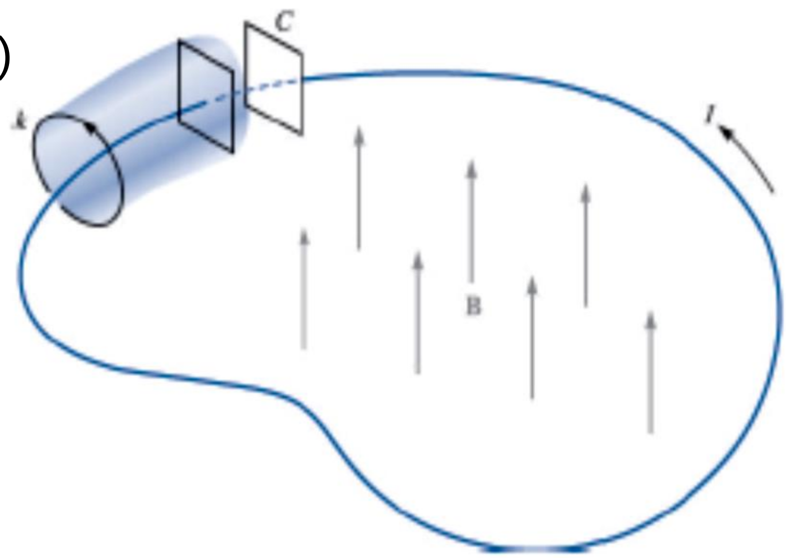


Fig. 4.3 A filamentary conductor forms a loop connecting the two parallel-plate capacitor.

4.3 Ampere's Law for Time varying Fields (Continued)

4.3.2 Maxwell's equation

- We may obtain the time-varying Ampere's circuital law by integrating Eqn. (4.18) over the surface S, we get:

$$\nabla x\vec{H} = \vec{J} + \vec{J}_d = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad (4.18)$$

$$\int_S (\nabla x\vec{H}).d\vec{S} = \int_S \vec{J}.d\vec{S} + \frac{\partial}{\partial t}\int_S \vec{D}.d\vec{S} \qquad (4.19)$$

Apply the Stokes' Theorem to the RHS of (4.19), we have:

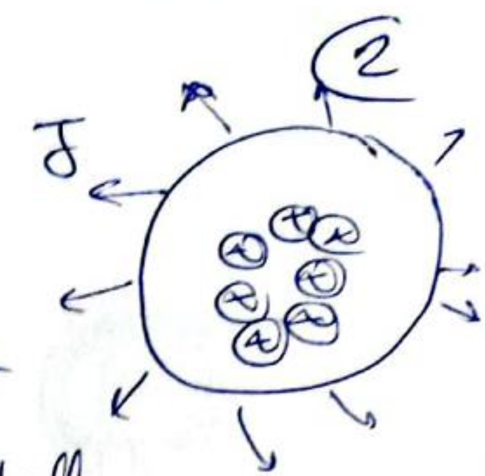
$$\oint_C \vec{H}.d\vec{l} = \int_S \vec{J}.d\vec{S} + \frac{\partial}{\partial t}\int_S \vec{D}.d\vec{S} \qquad (4.20)$$

Notes:

1. Maxwell had added an additional term ($\partial D/\partial t$) to the RHS of Ampere's law.
2. This new term removes the contradiction between Ampere's law and the continuity equation .
3. Eqns. (4.18) and (4.20) represent Ampere's Circuital Law or one of the four Maxwell's equations in both differential and integral form.

Summary of 4th Maxwell equations

2nd equation



As shown; if a current flow in a Region (Volume), This Current will generate current density on the surface of charge within the Region itself & outward current through enclosing surface can be equal to $\oint\oint \bar{J} \cdot \bar{ds}$

كلما كانت هناك شحنات داخل منطقة مقفوله فإن الشحنات عندما تتحرك تعمل تيار I والتيار لو اقترب من السطح يعمل J وكلما قلت الشحنات حول السطح زادت J خارج السطح.

if Q_m = Total charge enclosed By Region (volume)

$$\therefore \quad Q_m = \iiint \rho_v \, dv$$

مجموع J على السطح = I كلما نقص معدل تغير الشحنة جوه الـ value كلما زاد التيار على السطح

$$\therefore \quad \oiint \bar{J} \cdot \bar{ds} = -\frac{\partial}{\partial t} \iiint \rho_v \, dv \quad [1]$$

معدل التغير ← $\frac{\partial}{\partial t}$

- from divergence Theorem (div)

تحويل من سطح لحجم

$$\oiint \bar{J} \cdot \bar{ds} = \iiint_v \nabla \cdot \bar{J} \, dv \quad [2]$$

From 1 & 2

$$\therefore \quad \nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t} \quad \rightarrow \text{Continuity equation}$$

- **From Ampere law**

$$\therefore \nabla \times \bar{H} = \bar{J}$$

نعوض div للطرفين

$$\nabla \cdot \nabla \times \bar{H} = zero$$

div

$$\text{But } \nabla \cdot \bar{J} \neq$$

نعمل
ماكسويل حلها

$$\because \nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}$$

$$\therefore \nabla \cdot \bar{J} + \frac{\partial \rho_v}{\partial t} = 0$$

$$\therefore \nabla \cdot \nabla \times \bar{H} = \left(\nabla \cdot \bar{J} + \frac{\partial \rho_v}{\partial t}\right) = 0$$

(3)

$$\nabla \cdot \nabla \times \bar{H} = \nabla \cdot \bar{J} + \frac{\partial \rho_v}{\partial t} = \nabla \cdot \bar{J} + \frac{\partial \nabla \cdot D}{\partial t}$$

$$\because \boxed{\rho_v = \nabla \cdot D}$$ معادلة ماكسويل في الـ static

$$\therefore \nabla \cdot \nabla \times \bar{H} = \nabla \cdot \bar{J} + \frac{\partial \nabla \cdot D}{\partial t} = \nabla \cdot \frac{\partial \bar{D}}{\partial t} + \nabla \cdot \bar{J}$$

$$\therefore \boxed{\nabla \times \bar{H} = \bar{J}_c + \frac{\partial \bar{D}}{\partial t}}$$

معادلة ماكسويل رقم 2 في Time varying

types of Current densities J_{cond} & J_d

$\nabla \times \bar{H} = \bar{J}_c + \bar{J}_d$ → dielectric current density $= (J_d)$

$\bar{J}_c$ → conduction current density $= (J_{cond.})$

* Conduction Current density (in conductors) $(J_{con} = \sigma \bar{E}$ & $I_c = \sigma E A)$

* Dielectric (displacement current density) (in capacitors)

$$J_d = \frac{\partial \bar{D}}{\partial t} = \epsilon \frac{\partial \bar{E}}{\partial t}$$

4.4 Generalized Maxwell's Equations

Maxwell's equations are derived on the basis of Ampere's circuital law, Faraday's law and Gauss's law both in point (differential) and integral form. These equations which are compiled by Maxwell led to the discovery of electromagnetic waves. For a field to be qualified as an electromagnetic field, it must satisfy all four of Maxwell's equations.

Table 4.1

Differential or Point Form	Integral Form	Remarks
$\nabla.\vec{D} = \rho_v$	$\oint_S \vec{D}.d\vec{s} = \int_{vol} \rho_v dv$	**Gauss's Law for Electrostatic Field**
$\nabla.\vec{B} = 0$	$\oint_S \vec{B}_S.d\vec{s} = 0$	**Gauss's Law for Magnetostatic Field** Nonexistence of magnetic charge
$\nabla x\vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\oint_L \vec{E}.d\vec{l} = -\frac{\partial}{\partial t}\int_S \vec{B}.d\vec{S}$	**Faraday's Law**
$\nabla x\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$	$\oint_C \vec{H}.d\vec{l} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right).d\vec{S}$	**Ampere's Circuital Law**

4.5 Time Harmonic Fields

- Let us assume that the electromagnetic fields are time-harmonic, i.e., they vary periodically or sinusoidally with time. So, the phasor form is applied to electromagnetic fields.
- A phasor is a complex number which can be written as $re^{j\phi}$ and $\phi = \omega t + \theta$ where θ may be function of time or space co-ordinates or constant.
- So, any vector field $\vec{A}$ which is sinusoidally time-varying can be represented as:

$$\vec{A}(t) = \vec{A}_o e^{j(\omega t+\theta)} \qquad (4.21)$$

The real part of $\vec{A}(t) = \vec{A}_o \cos(\omega t + \theta)$

and imaginary part of $\vec{A}(t) = \vec{A}_o \sin(\omega t + \theta)$

- While performing mathematical expressions, either the real part or the imaginary part of a quantity should be used but not both at same time.
- By dropping the time factor $e^{j\omega t}$ in $\vec{A}(t)$, the resulting complex term $\vec{A}_o e^{j\theta}$ is called the phasor of vector field and is denoted by $\vec{A}_s$ i.e.

$$\vec{A}_s = \vec{A}_o e^{j\theta} \qquad (4.22)$$

where subscript s denotes the phasor form of $\vec{A}(t)$.

- Thus $\vec{A}(t) = \vec{A}_o \cos(\omega t + \theta)$ is the instantaneous form and it can be expressed

4.5 Time Harmonic Fields (Continued)

as: $$\vec{A}(t) = \mathrm{Re}\left(\vec{A}_s e^{j\omega t}\right)$$

so $$\frac{\partial \vec{A}(t)}{\partial t} = \frac{\partial}{\partial t}\left(\mathrm{Re}\,\vec{A}_s e^{j\omega t}\right) = \mathrm{Re}\left(j\omega \vec{A}_s e^{j\omega t}\right) = j\omega \vec{A}_s \qquad (4.23)$$

- Eqn. (4.23) shows that the time derivative of instantaneous quantity is equivalent to multiplying its phasor form by jω. Similarly,

$$\int \vec{A}(t)dt = \vec{A}_s / j\omega \qquad (4.24)$$

- So, the Maxwell's equations for time-varying fields are modified and given in Table 4.2. All the field quantities and their derivatives can be expressed in phasor form using eqn. (4.23) and eqn. (4.24).

Table 4.2

4.5 Time Harmonic Fields (Continued)

Example 4.4

Express the following vectors in phasor form:

a) $E_y(z, t) = 100 \cos(10^8 t - 0.5z + 30^o)$ V/m

b) $\vec{A}(x,t) = 10\cos(10^8 t - 10x + 60^o)\ \hat{a}_z$

Solution:

a) Step #1: Write the exponential notation or equation as:

$$E_y(z, t) = \mathrm{Re}[100\, e^{j(10^8 t - 0.5z + 30^o)}]$$

Step #2: drop Re and suppress $e^{j10^8 t}$ to obtaining the phasor:

$$E_{ys}(z) = 100\, e^{(-j\,0.5z + j\,30^o)}$$

b) Similarly:

$$\vec{A}(x,t) = \mathrm{Re}[10e^{j(-10x+60^o)}e^{j\omega t}]\ \hat{a}_z = \mathrm{Re}[A_s e^{j\omega t}]\ \hat{a}_z$$

and $\omega = 10^8$. Hence, we get:

or $$\vec{A}_s(x) = 10e^{j(-10x+60^o)}\ \hat{a}_z$$

4.5 Time Harmonic Fields (Continued)

Example 4.5

Express the following vector in instantaneous form:

$$\vec{B}_s(x) = (20/j)\hat{a}_x + 10e^{j2\pi x/3}\hat{a}_y$$

Solution:

$$\vec{B}_s(x) = (20/j)\hat{a}_x + 10e^{j2\pi x/3}\hat{a}_y = -j20\hat{a}_x + 10e^{j2\pi x/3}\hat{a}_y$$

$$= 20e^{-j\pi/2}\hat{a}_x + 10e^{j2\pi x/3}\hat{a}_y$$

Then the instantaneous form is:

$$\vec{B}(x,y,t) = \mathrm{Re}(\vec{B}_s e^{j\omega t}) = \mathrm{Re}(20e^{j(\omega t-\pi/2)}\hat{a}_x + 10e^{j(\omega t+2\pi x/3)}\hat{a}_y)$$

$$= 20\cos(\omega t - \pi/2)\hat{a}_x + 10\cos(\omega t + 2\pi x/3)\hat{a}_y$$

$$= 20\sin(\omega t)\hat{a}_x + 10\cos(\omega t + 2\pi x/3)\hat{a}_y$$

Thank you for your attention